On Nonlocality of p-Adic and Zeta Strings

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1. Introduction: motivation to study *p*-adic strings

MANY REASONS TO STUDY *p*-ADIC STRINGS:

- *p*-adic strings have *p*-adic valued world sheet
- *p*-adic strings are related to ordinary strings
- *p*-adic strings are simpler than ordinary strings
- *p*-adic strings have exact Lagrangian
- *p*-adic strings have nonlinear and nonlocal dynamics
- *p*-adic strings have a non-Archimedean (ultrametric) structure

1. Introduction

- I will give a brief review of basic properties of p-adic strings.
- I will consider a model with possible p-adic matter in FLRW universe with Einstein-Hilbert action.
- Some aspects of p-adic strings sector will be presented as zeta strings.

2. p-Adic Strings

Volovich, Vladimirov, Freund, Witten, Arefeva, B.D., ...
String amplitudes:

- standard crossing symmetric Veneziano amplitude

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} \, |1-x|_\infty^{b-1} \, d_\infty x$$
$$= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}$$

- p-adic crossing symmetric Veneziano amplitude

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} \, |1-x|_p^{b-1} \, d_p x$$
$$= g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

2. p-Adic Strings

- Freund-Witten product formula for adelic strings

$$A(a,b) = A_\infty(a,b)\prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = const.$$

3. Effective Field Theory for p-Adic Strings

- One of the greatest achievements in p-adic string theory is an effective field description of scalar open and closed p-adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher (Koba-Nielsen) ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m_p^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big]$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and metric with signature $(- + ... +)$ (Freund, Witten, Frampton, Okada, ...) .

The above Lagrangian is written completely in terms of real numbers and there is no explicit dependence on the world sheet. However, it can be rewritten as:

$$\mathcal{L}_p = \frac{m^D}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{2}\,\varphi\int_{\mathbb{R}}\Big(\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}\chi_p(u)|u|_p^{\frac{k^2}{2m^2}}\,du\Big)\tilde{\varphi}(k)\,\chi(kx)\,d^4k + \frac{1}{p+1}\,\varphi^{p+1}\Big],$$

where $\chi(kx) = e^{-2\pi ikx}$ is the real additive character. Since $\int_{\mathbb{Q}_p}\chi_p(u)|u|^{s-1}du = \frac{1-p^{s-1}}{1-p^{-s}} = \Gamma_p(s)$ and it is present in the scattering amplitude, one can say that $\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}\chi_p(u)|u|_p^{\frac{k^2}{2m^2}}du$ is related to the p-adic string world-sheet.

3. Effective Field Theory for p-Adic Strings

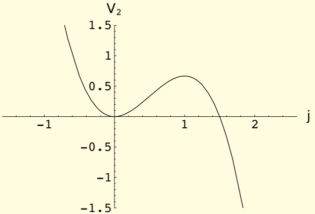

Figure: The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the left) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the right)

Potential .

$$\mathcal{V}_p(\varphi) = \frac{m_p^D}{g_p^2}\left[\frac{1}{2}\frac{p^2}{p-1}\varphi^2 - \frac{p^2}{p^2-1}\varphi^{p+1}\right].$$

3. Effective Field Theory for p-Adic Strings

- The equation of motion is

$$p^{-\frac{\Box}{2m^2}}\,\varphi = \varphi^p, \quad \varphi = 0,\ \varphi = 1,\ (\varphi = -1,\ p \neq 2)$$

$$e^{A\partial_t^2}\ e^{Bt^2} = \frac{1}{\sqrt{1-4AB}}\,e^{\frac{Bt^2}{1-4AB}}, \quad 1 - 4AB > 0$$

There are also nontrivial solutions:.

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2m^2\,p\ln p}(x^i)^2\Big)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}}\ \exp\Big(\frac{p-1}{2p\ln p}m^2 t^2\Big)$$

$$\varphi(x) = p^{\frac{D}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2\,p\ln p}\,m^2\,x^2\Big), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2 .$$

3. Effective Field Theory for *p*-Adic Strings

- Prime number p can be replaced by natural number $n \geq 2$ and such expression also makes sense. Moreover, when $p = 1 + \varepsilon \to 1$ there is the limit which is related to the ordinary bosonic string in the boundary string field theory (Gerasimov-Shatashvili):

$$\mathcal{L} = \frac{m^D}{g^2} \Big[\frac{1}{2}\, \varphi\, \frac{\Box}{m^2}\, \varphi + \frac{\varphi^2}{2}\, (\ln \varphi^2 - 1)\Big]$$

- From these and many other developments it follows that some nontrivial features of ordinary strings are similar to p-adic ones and are related to the p-adic effective action.

4. Possible p-Adic Matter

To avoid tachyon, consider transition $m^2 \to -m^2$ in D = 4 dimensions. Also change sign to lagrangian to avoid ghost. Then the related new Lagrangian is

$$L_p = -1\,\frac{m^4}{g^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\phi\,p^{\frac{\Box}{2m^2}}\,\phi + \frac{1}{p+1}\,\phi^{p+1}\Big] \qquad (1)$$

with the corresponding potential

$$V_p(\phi) = (-1)\,\frac{m^D}{g^2}\,\Big[\frac{1}{2}\,\frac{p^2}{p-1}\,\phi^2 - \frac{p^2}{p^2-1}\,\phi^{p+1}\Big].$$

and equation of motion

$$p^{\frac{\Box}{2m^2}}\,\phi_p = \phi_p^p \qquad (2)$$

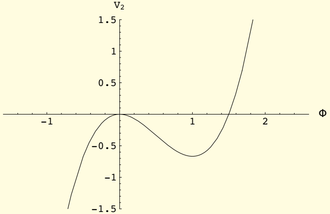

Figure: New potentials $V_2(\phi)$ and $V_3(\phi)$, which are related to new Lagrangian.

Trivial solutions

$$p^{\frac{\Box}{2m^2}}\,\phi = \phi^p, \quad \phi = 0,\ \phi = 1,\ (\phi = -1,\ p \neq 2)$$

4. Possible p-Adic Matter

Consider field ϕ around minimum $\phi = 1 + \theta$. Then EOM for weak field θ, i.e $\theta_p \ll 1$, becomes

$$p^{\frac{\Box}{2m^2}}\,(1+\theta_p) = (1+\theta_p)^p, \quad \Rightarrow \quad p^{\frac{\Box}{2m^2}}\,\theta_p = p\,\theta_p$$

Explore dynamics of $\theta_p(t)$ in FLRW metric with constant Hubble parameter. Let corresponding KG equation is

$$\Box\theta_p = 2m^2\theta_p, \qquad \Box = -\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t}, \quad H = const. \qquad (3)$$

Suppose a solution in the form $\theta_p(t) = Ce^{\lambda t}$. Then the above KG equation has solution $\theta_p(t) = Ce^{-mt}$ with $H = m$, that is when scale factor is $a(t) = Ae^{mt}$. One can easily check that $\theta_p(t) = Ce^{-mt}$ satisfies EOM $p^{\frac{\Box}{2m^2}}\,\theta_p = p\,\theta_p$.

4. Possible p-Adic Matter

Suppose $a(t) = Ae^{mt}$ is scale factor for closed universe (k=+1) of the gravity model with nonlocal scalar field θ_p. The corresponding action and EOM are ($\gamma = \frac{1}{16\pi G}$ and $\sigma = -\frac{m^4}{g^2}\frac{p^2}{p-1}$.):

$$S = \gamma \int \sqrt{-g}\, d^4x\, (R - 2\Lambda) + \sigma_p \int \sqrt{-g}\, d^4x \left(-\frac{1}{2}\theta_p\, p^{\frac{1}{2m^2}\Box}\, \theta_p + \frac{p}{2}\, \theta_p^2 \right),$$

$$\gamma(G_{\mu\nu} + \Lambda g_{\mu\nu}) + \frac{\sigma_p}{4}\Big[g_{\mu\nu}\, \theta_p p^{\frac{\Box}{2m^2}} \theta_p - g_{\mu\nu}\, p\, \theta_p^2 - \Omega_{\mu\nu}(\theta_p) \Big] = 0,$$

$$p^{\frac{\Box}{2m^2}}\, \theta_p = p\, \theta_p .$$

$$\Omega_{\mu\nu}(\theta_p) = \sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1} \Big[g_{\mu\nu}\, \left(\nabla^\alpha \Box^\ell \theta_p \nabla_\alpha \Box^{n-1-\ell}\theta_p + \Box^\ell \theta_p \Box^{n-\ell}\theta_p\right) - 2\nabla_\mu \Box^\ell \theta_p \nabla_\nu \Box^{n-1-\ell}\theta_p \Big].$$

4. Possible p-Adic Matter

Now we have

$$\gamma(G_{\mu\nu} + \Lambda g_{\mu\nu}) - \frac{\sigma_p}{4}\,\Omega_{\mu\nu}(\theta_p) = 0.$$

Two linearly independent equations:

$$\gamma(4\Lambda - R) - \frac{3\sigma_p}{4}\,p\ln p\,\theta_p^2 = 0,$$
$$\gamma(3m^2 + \frac{3}{A^2C^2}\theta_p^2 - \Lambda) + \frac{3\sigma_p}{8}\,p\ln p\,\theta_p^2 = 0,$$

where $R = 12m^2 + \frac{6}{A^2C^2}\theta_p^2$

4. Possible p-Adic Matter

Finally

$$\theta_p(t) = Ce^{-mt}, \qquad a(t) = Ae^{mt}, \qquad k = +1$$

for

$$\Lambda = 3m^2, \qquad \frac{1}{A^2 C^2} = 2\pi G \frac{m^4}{g^2} \frac{p^3}{p-1} \ln p.$$

5. Lagrangians for p-Adic String Sector

Recall

- scattering amplitude for a p-adic string

$$A_p(a,b,c) = g_p^2 \, \frac{1-p^{a-1}}{1-p^{-a}} \, \frac{1-p^{b-1}}{1-p^{-b}} \, \frac{1-p^{c-1}}{1-p^{-c}}$$

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \, \frac{p^2}{p-1} \Big[-\frac{1}{2} \, \varphi \, p^{-\frac{\Box}{2m_p^2}} \, \varphi + \frac{1}{p+1} \, \varphi^{p+1} \Big]$$

- scattering amplitude for p-adic string sector

$$\prod_p A_p(a,b,c) = \prod_p g_p^2 \, \frac{\zeta(a)}{\zeta(1-a)} \, \frac{\zeta(b)}{\zeta(1-b)} \, \frac{\zeta(c)}{\zeta(1-c)}$$

Lagrangian for p-adic sector = ?

5. Lagrangians for p-Adic String Sector

This Lagrangian should contain the Riemann zeta function.

Recall definition of the Riemann zeta function:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \;\; \sigma > 1$$

Two approaches: multiplicative and additive.

5. Lagrangians for p-Adic String Sector: multiplicative approach

Approach, which is based on a multiplication of some parts of p-adic Lagrangian. Riemann zeta function emerges through its product form. Our starting point is p-adic Lagrangian

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \Big[-\frac{1}{2} \varphi\, p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \Big]$$

with $m_p^2 = m^2$ for every p.
It is useful to rewrite Lagrangian in the form

$$\mathcal{L}_p = \frac{m^D}{g_p^2} \frac{p^2}{p^2-1} \Big\{ \frac{1}{2} \varphi \Big[\Big(1 - p^{-\frac{\Box}{2m^2}+1} \Big) + \Big(1 - p^{-\frac{\Box}{2m^2}} \Big) \Big] \varphi \\ - \varphi^2 \Big(1 - \varphi^{p-1} \Big) \Big\}$$

5. Lagrangians for p-Adic String Sector: multiplicative approach

$$\mathcal{L} = m^D \prod_p \mathcal{L}'_p$$

$$\prod_p g_p^2 = C\,, \quad \prod_p \frac{1}{1 - p^{-2}}\,, \quad \prod_p \left(1 - p^{-\frac{\Box}{2m^2}+1}\right),$$

$$\prod_p \left(1 - p^{-\frac{\Box}{2m^2}}\right), \quad \prod_p \left(1 - \varphi^{p-1}\right)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}\,, \qquad s = \sigma + i\tau\,, \quad \sigma > 1$$

5. Lagrangians for p-Adic String Sector: multiplicative approach

New Lagrangian becomes

$$\mathcal{L} = \frac{m^D}{C}\,\zeta(2)\left\{\frac{1}{2}\,\phi\Big[1/\zeta\Big(\frac{\Box}{2m^2} - 1\Big) + 1/\zeta\Big(\frac{\Box}{2m^2}\Big)\Big]\,\phi - \phi^2 G(\phi)\right\}$$

with $G(\phi) = \mathcal{AC}\prod_p(1-\phi^{p-1})$, where $\mathcal{AC}$ denotes analytic continuation of infinite product $\prod_p(1-\phi^{p-1})$, which is convergent if $|\phi|_\infty < 1$. One can easily see that $G(0) = 1$ and $G(1) = G(-1) = 0$.

It is worth noting two interesting possibilities for the coupling constant g_p: (1) $g_p^2 = \frac{p^2}{p^2-1}$, what yields $\zeta(2)/C = 1$, and (2) $g_p = |r|_p$, where r may be any non zero rational number and it gives $|r|_\infty \prod_p |r|_p = 1$. Both these possibilities are consistent with adelic product formula. For simplicity, in the sequel we shall take $C = \zeta(2)$.

5. Lagrangians for p-Adic String Sector: multiplicative approach

The corresponding equation of motion is

$$\Big[1/\zeta\Big(\frac{\Box}{2m^2} - 1\Big) + 1/\zeta\Big(\frac{\Box}{2m^2}\Big)\Big]\,\phi = 2\phi\, G(\phi) + \phi^2\, G'(\phi)$$

and has $\phi = 0$ as a trivial solution. In the weak-field approximation ($\phi(x) \ll 1$), equation becomes

$$\Big[1/\zeta\Big(\frac{\Box}{2m^2} - 1\Big) + 1/\zeta\Big(\frac{\Box}{2m^2}\Big)\Big]\,\phi = 2\phi$$

$\zeta\Big(\frac{\Box}{2m^2}\Big)$ can be regarded as a pseudodifferential operator

$$1/\zeta\Big(\frac{\Box}{2m^2}\Big)\,\phi(x) = \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} e^{ixk}\, 1/\zeta\Big(-\frac{k^2}{2m^2}\Big)\,\tilde{\phi}(k)\,dk$$

5. Lagrangians for p-Adic String Sector: multiplicative approach

Mass spectrum of M^2 is determined by solutions of equation

$$1/\zeta\Big(\frac{M^2}{2m^2} - 1\Big) + 1/\zeta\Big(\frac{M^2}{2m^2}\Big) = 2$$

There are infinitely many tachyon solutions, which are below the largest one $M^2 \approx -3.5m^2$.

5. Lagrangians for p-Adic String Sector: multiplicative approach

The potential follows from $-\mathcal{L}$ at $\Box = 0$, i.e.

$$V(\phi) = m^D \,[7 + G(\phi)]\, \phi^2$$

since $\zeta(-1) = -1/12$ and $\zeta(0) = -1/2$. This potential has local minimum $V(0) = 0$ and values $V(\pm 1) = 7\, m^D$. To explore behavior of $V(\phi)$ for all $\phi \in \mathbb{R}$ one has first to investigate properties of the function $G(\phi)$.

5. Lagrangians for p-Adic String Sector: additive approach

It is worth noting that a Lagrangian similar to the above one can be obtained by an additive approach. Namely,

$$L = \sum_{n=1}^{+\infty} C_n \, \mathcal{L}_n = \frac{m^D}{2} \, \phi \Big[\sum_{n=1}^{+\infty} \mu(n) \, n^{-\frac{\Box}{2m^2}+1} + \sum_{n=1}^{+\infty} \mu(n) \, n^{-\frac{\Box}{2m^2}} \Big] \phi - m^D \sum_{n=1}^{+\infty} \mu(n) \, \phi^{n+1}$$

where $\frac{C_n}{g_n^2} \frac{n^2}{n-1} = D_n = -\mu_n(n+1), \; n = 1, 2, ...$ and $\mu(n)$ is the Möbius function:

$$\mu(n) = \begin{cases} 0, & n = p^2 m \\ (-1)^k, & n = p_1 p_2 \cdots p_k, \; p_i \neq p_j \\ 1, & n = 1, \; (k = 0) \end{cases} \tag{4}$$

5. Lagrangians for p-Adic String Sector: additive approach

Introducing zeta function by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

one can rewrite Lagrangian in the form

$$L = m^D \Big\{ \frac{1}{2}\,\phi \Big[1/\zeta(\frac{\Box}{2m^2} - 1) + 1/\zeta(\frac{\Box}{2m^2}) \Big]\,\phi - \phi^2\, F(\phi) \Big\}$$

where $F(\phi) = \mathcal{AC} \sum_{n=1}^{+\infty} \mu(n)\, \phi^{n-1}$.

5. Lagrangians for p-Adic String Sector: additive approach

The difference between Lagrangians $\mathcal{L}$ (multiplicative approach) and L (additive approach) is only in functions $G(\phi)$ and $F(\phi)$. Since

$$G(\phi) = \prod_p \left(1 - \phi^{p-1}\right) = 1 - \phi - \phi^2 + \phi^3 - \phi^4 + \dots$$

and

$$F(\phi) = \sum_{n=1}^{\infty} \mu(n) \phi^{n-1} = 1 - \phi - \phi^2 - \phi^4 + \dots$$

it follows that these functions have the same behavior for $|\phi| \ll 1$. Hence, in weak-field approximation these Lagrangians describe the same scalar field theory.

5. Lagrangians for p-Adic String Sector: additive approach

Another example is based on

$$\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n^s} = (1 - 2^{1-s})\,\zeta(s), \quad s = \sigma + i\tau\,, \quad \sigma > 0$$

which has analytic continuation to the entire complex s plane without singularities. The corresponding Lagrangian is

$$L = m^D\Big[-\frac{1}{2}\phi\Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi + \phi - \frac{1}{2}\,\log(1+\phi)^2\Big].$$

5. Lagrangians for p-Adic String Sector: additive approach

The potential is

$$V(\phi) = -L(\Box = 0) = m^D \Big[\frac{1}{4}\,\phi^2 - \phi + \frac{1}{2}\log(1+\phi)^2\Big],$$

which has one local maximum $V(0) = 0$ and one local minimum at $\phi = 1$. It is singular at $\phi = -1$, i.e. $V(-1) = -\infty$, and $V(\pm\infty) = +\infty$. The equation of motion is

$$\Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi = \frac{\phi}{1+\phi},$$

which has two trivial solutions: $\phi = 0$ and $\phi = 1$.

6. Concluding Remarks

- *p*-Adic strings are nonlocal, nonlinear and non-Archimedean objects, which are in many ways related to ordinary strings.
- There is an example of *p*-adic matter.
- Constructed new Lagrangians for *p*-adic strings sector contain Riemann zeta function nonlocality.
- There is a sense to continue with developments of *p*-adic and zeta strings, as well as *p*-adic matter.

Some references on zeta strings

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THANK YOU FOR YOUR ATTENTION!