# Ghosts in ghost-free analytic infinite derivative gravity theories

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**Mainly based on recent papers**

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# Introduction

## Grand Problem

- Einstein's gravity is not UV-complete

## Interesting development

- Stelle's 1977 and 1978 papers show that $R^2$ gravity is renormalizable gravity with the price of a physical (Weyl) ghost

## Big success

- Starobinsky inflation is based on $R^2$ and works perfectly

## Important keywords and names to mention

- Form-factors, String field theory, . . . ;
  Aref'eva, Barvinsky, Biswas, Efimov, Koivisto, Krasnikov, Kuz'min, Mazumdar, Modesto, Sen, Shapiro, Siegel, Tomboulis, Witten, Zwiebach, . . .

## Pure gravity arguments why infinite derivatives are crucial

We start with

$$S = \int d^D x \sqrt{-g} \left( \mathcal{P}_0 + \sum_i \mathcal{P}_i \prod_I (\hat{\mathcal{O}}_{iI} \mathcal{Q}_{iI}) \right)$$

Here $\mathcal{P}$ and $\mathcal{Q}$ depend on curvatures and $\mathcal{O}$ are operators made of covariant derivatives.

Everywhere the respective dependence is *analytic* in IR.

Let's name it *general analytic gravity*

## Action to study linear perturbations around MSS

The result is [arxiv:1602.08475, arXiv:1606.01250]

$$S = \int d^D x \sqrt{-g} \left( \frac{M_P^2 R}{2} - \Lambda \right.$$

$$\left. + \frac{\lambda}{2} \left( R \mathcal{F}_R(\Box) R + L_{\mu\nu} \mathcal{F}_L(\Box) L^{\mu\nu} + W_{\mu\nu\lambda\sigma} \mathcal{F}_W(\Box) W^{\mu\nu\lambda\sigma} \right) \right)$$

Here $\mathcal{F}_X(\Box) = \sum_{n \geq 0} f_{X_n} \Box^n$ and $L_{\mu\nu} = R_{\mu\nu} - \frac{1}{D} R g_{\mu\nu}$

Thanks to the Bianchi identities one can further achieve $\mathcal{F}_L(\Box) = 0$ in $D = 4$ and $\mathcal{F}_L(\Box) = \mathrm{const}$ in $D > 4$.

## Spin-2:

$$S_2 = \frac{1}{2}\int dx^4 \sqrt{-\bar{g}}\, h^{\perp}_{\nu\mu} \left(\bar{\Box} - \frac{\bar{R}}{6}\right) [\mathcal{P}(\bar{\Box})]\, h^{\perp\,\mu\nu}$$

$$\mathcal{P}(\bar{\Box}) = 1 + \frac{2}{M_P^2}\lambda f_{R_0}\bar{R} + \frac{2}{M_P^2}\lambda \mathcal{F}_W \left(\bar{\Box} + \frac{\bar{R}}{3}\right)\left(\bar{\Box} - \frac{\bar{R}}{3}\right)$$

**The Stelle's case corresponds to $\mathcal{F}_W = 1$ such that**

$$\mathcal{P}(\bar{\Box})_{Stelle} = 1 + \frac{2}{M_P^2}\lambda f_{R_0}\bar{R} + \frac{2}{M_P^2}\lambda \cdot 1 \cdot \left(\bar{\Box} - \frac{\bar{R}}{3}\right)$$

**This is an obvious second pole which will be the ghost.**

**Spin-0:**

$$S_0 = -\frac{1}{2}\int dx^4 \sqrt{-\bar{g}}\ \phi(3\bar{\Box} + \bar{R})\left[\mathcal{S}(\bar{\Box})\right]\phi$$

$$\mathcal{S}(\bar{\Box}) = 1 + \frac{2}{M_P^2}\lambda f_{R_0}\bar{R} - \frac{2}{M_P^2}\lambda \mathcal{F}_R(\bar{\Box})(3\bar{\Box} + \bar{R})$$

**This *is* the ghost in Einstein-Hilbert case $\mathcal{F}_R = 0$, but it is constrained and is not physical.**

**Thus, $\mathcal{S}(\bar{\Box})$ *can* have one root as a function of $\bar{\Box}$ and as such generate one more pole in the propagator and it will be not a ghost. That is, $\mathcal{F}(\bar{\Box}) = \text{const}$**

**This would be exactly the scalar mode in a local $f(R)$ gravity.**

## Things to mention

- **Ostrogradski statement from 1850 forbids higher derivatives. The Weyl tensor already has 2, its square has 4 and constraints do not alleviate the problem. This is why the Weyl ghost arises.**
- **Ostrogradski statement does not tell about infinite number of derivatives.**
- **Some theories like Horndeski, for example, can avoid ghosts with a finite number of derivatives but their heavy fine-tuning raises concerns.**
- **String Field Theory (SFT) provides a consistent model where higher (infinite) derivatives are present.**

## Physical excitations

Effectively we modify the propagators as follows

$$\Box - m^2 \to \mathcal{G}(\Box)$$

Recall, in $D = 4$ in $(- + + +)$

$$L = \frac{1}{2}\phi(\Box - m^2)\phi \text{ – good field}$$

$-\Box$ gives a ghost, $+m^2$ gives a tachyon (for real $m$).

Consider

$$L = \frac{1}{2}\phi(\Box - m^2)(\Box - \mu^2)\phi$$

This Lagrangian describes 2 physical excitations and the second one is a ghost. The higher degree polynomial in $\Box$ will just produce more ghosts.

## Analytic Infinite Derivative (AID) way around

To preserve the physics we demand

$$\mathcal{G}(\Box) = (\Box - m^2)e^{2\sigma(\Box)}$$

where $\sigma(\Box)$ must be an *entire* function resulting in the fact that the exponent of it has no roots.

Thus

$$L = \frac{1}{2}\phi(\Box - m^2)e^{2\sigma(\Box)}\phi$$

So, yes, we can incorporate infinite number of derivatives by employing properties of entire functions.

## String Field Theory point of view

**Going into the SFT computations one can deduce Lagrangians of the following form**

$$L = \varphi(\Box - m^2)\varphi - \lambda(e^{\rho(\Box)}\varphi)^4$$

**There is a transparent field redefinition $\phi = e^{\rho(\Box)}\varphi$ resulting in**

$$L = \phi(\Box - m^2)e^{-2\rho(\Box)}\phi - \lambda\phi^4$$

**Obviously, we arrive to the kinetic term from the previous slide identifying $\sigma(\Box) = -\rho(\Box)$**

## Entire functions

- **An entire function is constant if it is analytic at infinity (Liouville theorem)**
- **An exponent of an entire function is again an entire function but *without zeroes* in the complex plane**
- **If a function has a pole at infinity, its Taylor series at zero in $w = 1/z$ must have finite number of terms**
- **An exponent of an entire function would have an infinite Taylor series at zero in $w = 1/z$ and this corresponds to the *essential singularity***
- **In any punctured neighbourhood of the essential singularity point the functions assumes all possible complex values, with perhaps a single exception, infinitely often (Picard theorem)**

**This all implies that our $\mathcal{G}(\Box)$ *must* have an essential singularity at infinity.**

## Spin-2 in details

Let's take $\Lambda = 0$. Then Minkowski is a solution while (A)dS - not. We continue with Minkowski and as such $\bar{R} = 0$.

$$S_2 = \frac{1}{2}\int dx^4\sqrt{-\bar{g}}\; h^{\perp}_{\nu\mu}\bar{\Box}\mathcal{P}(\bar{\Box})h^{\perp\mu\nu}$$

$$\mathcal{P}(\bar{\Box}) = 1 + \frac{2}{M_P^2}\lambda\bar{\Box}\mathcal{F}_W\left(\bar{\Box}\right) = e^{2\omega(\Box)}$$

$\omega(\Box)$ is an entire function and $\omega(0) = 0$ to have correct IR limit.

The Stelle's proposal is improved by inserting an obligatory infinite derivative form-factor and no algebraic way around is seen.

We have no ghosts around Minkowski background but it is not the proof of unitarity yet.

Can this be a quantum gravity?

## Other backgrounds including Starobinsky inflation

For any (and in most cases only if):

$$\Box R = r_1 R + r_2$$

We have a solution if (here $\mathcal{F} \equiv \mathcal{F}_R$):

$$\mathcal{F}^{(1)}(r_1) = 0, \ \frac{r_2}{r_1}(\mathcal{F}(r_1) - f_0) = -\frac{M_P^2}{2\lambda} + 3r_1\mathcal{F}(r_1),$$
$$4\Lambda r_1 = -r_2 M_P^2, \text{ so when } \Lambda = 0 \Rightarrow r_2 = 0$$

If we stick to conformally flat backgrounds then the Weyl part does not contribute.

– Any solution of a local $R^2$ gravity can be a solution here

– Nothing else can be a background if you demand that eigenmodes of the d'Alembertian form a complete set of functions on your manifold [arXiv:1711.08864]

## Ghosts strike back

- **We found a generic model around MSS**
- **It was proven that at least spin-2 part must have a non-local form-factor**
- **The model is made ghost-free around Minkowski, only . . .**
- $\mathcal{F}(\Box)$ **fixed around one background will not exorcise ghosts around other backgrounds, pretty obviously, . . .**

**– We can assume that the word “quantum” is premature here.**

**– We can see whether ghosts are benign or not.**

## Non-local Higgs inflation as a toy model [arxiv:2006.06641]

The bottom-line AID modified action is as follows:

$$L = \frac{1}{2}M_P^2 R_E + \frac{1}{2}\phi\Box e^{2\sigma(\Box)}\phi - V(\phi)$$

$\sigma(\Box)$ is an entire function

We can make $\phi = 0, \infty$ to be ghost-free vacua but all the way in between effective new modes appear. Namely, this depends on algebraic roots of an equation

$$\Box\, e^{2\sigma(\Box)} = \frac{\partial^2 V(\phi)}{\partial\phi^2}$$

Choosing the potential we may have several points where its second derivative vanishes. For all other values $\phi$ we have infinitely many new effective modes.

## What are these new modes? – Half of them are ghosts!

- As long as the second derivative of the potential is non-zero there is an infinite number of new modes with complex conjugate masses squared and all are heavy with $|m| > M_P$
- The following condition
$$(\mathrm{Im}(m^2))^2 < 9H^2\mathrm{Re}(m^2)$$
guarantees no classical growing behavior for these new effective modes in an (A)dS space-time characterized by the Hubble rate $H$.
- It is important to understand that values of $m$ are governed mainly by the shape of the entire function and also by the value of $H$ originating from the potential while the restriction which excludes growing classical behavior does not depend on the entire function.

## Conclusions and Outlook

- A class of analytic infinite derivative (AID) theories has been considered targeting the goal of constructing a UV complete and unitary gravity
- It features many nice properties, like native embedding of the Starobinsky inflation, finite Newtonian potential at the origin, presence of a non-singular bounce, healing of non-renormalizable models including Higgs inflation, etc.
- The theory has clear connection to SFT
- BUT this approach is either incomplete to start building quantum gravity or ghosts, even though benign, must be accepted and met with dignity

**Thank you for listening!**