Black Holes in Higher Derivative Gravity

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Classical gravity with higher derivatives

Consider the gravitational action

$$I = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)\,.$$

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} \;\;=\;\; & \gamma\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{2}{3}\left(\alpha - 3\beta\right)\nabla_\mu\nabla_\nu R - 2\alpha\Box R_{\mu\nu} \\ & + \frac{1}{3}\left(\alpha + 6\beta\right)g_{\mu\nu}\Box R - 4\alpha R^{\eta\lambda}R_{\mu\eta\nu\lambda} + 2\left(\beta + \frac{2}{3}\alpha\right)RR_{\mu\nu} \\ & + \frac{1}{2}g_{\mu\nu}\left(2\alpha R^{\eta\lambda}R_{\eta\lambda} - \left(\beta + \frac{2}{3}\alpha\right)R^2\right) = \frac{1}{2}T_{\mu\nu} \end{aligned}$$

By expanding the action of this theory about flat space, one deduces the dynamical content of the linearised theory: *positive-energy* massless spin-two, *negative-energy* massive spin-two with mass $m_2 = \gamma^{\frac{1}{2}}(2\alpha)^{-\frac{1}{2}}$ and *positive-energy* massive spin-zero with mass $m_0 = \gamma^{\frac{1}{2}}[6\beta]^{-\frac{1}{2}}$. K.S.S. 1978

Static and spherically symmetric solutions

Consider what happens to spherically symmetric static solutions in the higher-curvature theory. One may choose to work in traditional Schwarzschild coordinates, for which the metric is given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In the linearised theory, one then finds the general solution to the source-free field equations $H^L_{\mu\nu} = 0$, where
$C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$ are six integration constants:

$$A(r) =$$

$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$
$$+\frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$

$$B(r) =$$

$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearised theory, the general static, spherically symmetric solution is a combination of a massless Newtonian $1/r$ potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made *via* standard $h^{\mu\nu}T_{\mu\nu}$ minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(\vec{x})$, for example, one finds

$$\begin{aligned} A(r) &= 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \\ B(r) &= 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \end{aligned}$$

with specific combinations of the Newtonian $1/r$ and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

Frobenius Asymptotic Analysis

Asymptotic analysis of the field equations near the origin leads to study of the *indicial equations* for behaviour as $r \to 0$. K.S.S. 1978
Let

$$\begin{aligned} A(r) &= a_{\tilde{s}} r^{\tilde{s}} + a_{\tilde{s}+1} r^{\tilde{s}+1} + a_{\tilde{s}+2} r^{\tilde{s}+2} + \cdots \\ B(r) &= b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \cdots \end{aligned}$$

and analyse the conditions necessary for the lowest-order terms in r of the field equations $H_{\mu\nu} = 0$ to be satisfied. This gives the following results, for the general α, β higher derivative theory:

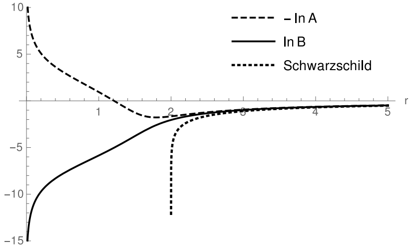
$$\begin{aligned} (\tilde{s}, t) &= (1, -1) \quad &&\text{with 5 free parameters} \\ (\tilde{s}, t) &= (0, 0) \quad &&\text{with 3 free parameters} \\ (\tilde{s}, t) &= (2, 2) \quad &&\text{with 6 free parameters} \end{aligned}$$

Lü, Perkins, Pope & K.S.S., 1508.00010

$(\tilde{s}, t) = (2, 2)$ solutions without horizons

For asymptotically flat solutions with nonzero spin-two Yukawa coefficient $C^{2-} \neq 0$, numerical solutions are found that can continue on in to mesh with the $(\tilde{s}, t) = (2, 2)$ family found by Frobenius asymptotic analysis around the origin. Some such solutions have no horizon; examples have been found numerically in the $m_2 = m_0$ theory B. Holdom, Phys. Rev. D66 (2002), hep-th 084010 and in the general $R - C^2 + R^2$ theory

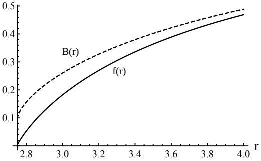
Lü, Perkins, Pope & K.S.S., 1508.00010; B. Holdom & J. Ren, 1612.04889 [gr-qc]



Horizonless solution in $R - C^2$ theory, behaving as r^2 in both $A(r)$ and $B(r)$ functions as $r \to 0$.

Wormholes

Another solution type found numerically has the character of a "wormhole". Such solutions can have either sign of $M \sim -C^{20}$ and either sign of the falling Yukawa coefficient C^{2-}. As an example, one finds a solution with $M < 0$ in the $R - \alpha C^2$ theory



In this solution, $f(r) = 1/A(r)$ reaches zero at a point where $B(r) = a_0^2 > 0$. Making a coordinate change $r - r_0 = \frac{1}{4}\rho^2$, one then has asymptotically as $r \to r_0$

$$ds^2 = -(a_0^2 + \tfrac{1}{4}B'(r_0)\rho^2)dt^2 + \frac{d\rho^2}{f'(r_0)} + (r_0^2 + \tfrac{1}{2}r_0\rho^2)d\Omega^2$$

which is $\mathbb{Z}_2$ symmetric in ρ and can be interpreted as a "wormhole", with the $r < r_0$ region excluded from spacetime.

Black hole solutions with horizons

If one assumes the existence of a horizon and assumes also asymptotic flatness at infinity, then a no-hair theorem for the trace of the field equations exists which implies that the Ricci scalar must vanish: $R = 0$. W. Nelson, 1010.3986; Lü, Perkins, Pope & K.S.S., 1508.00010 This significantly simplifies the analysis of the solutions. The field equations then become identical to those in the $\beta = 0$ case, *i.e.* with just a $(\text{Weyl})^2$ term and no R^2 term in the action.

Counting parameters in an expansion around the horizon, subject to the $R = 0$ condition, one finds just 3 free parameters. The Schwarzschild solution, satisfying $R_{\mu\nu} = 0$, is just such an asymptotically flat solution with a horizon, and it is characterised by two parameters: the mass M of the black hole, plus a trivial g_{00} normalisation parameter at infinity. So in the higher-derivative theory, there is just one extra "non-Schwarzschild" (NS) parameter relevant to the family of asymptotically flat solutions with a horizon.

Non-Schwarzschild (NS) Black Holes

Lü, Perkins, Pope & K.S.S., 1508.00010; 1502.01028

Now the question arises: what happens when one moves a finite distance away from a Schwarzschild solution in terms of the NS parameter? In general, one expects the solution to violate asymptotic flatness at spatial infinity for small deviations from Schwarzschild. But what about increasing the non-Schwarzschild parameter further? Does the loss of asymptotic flatness persist, or does something else happen, with solutions arising that cannot be treated by a linearised analysis in deviation from Schwarzschild?

This can be answered numerically. In consequence of the trace no-hair theorem, the assumption of a horizon together with asymptotic flatness requires $R = 0$ for the solution, so the calculations can effectively be done in the $R - \alpha C^2$ theory with $\beta = 0$, in which the field equations, thankfully, can be reduced to a system of two second-order equations.

The study of NS solutions is more easily carried out with a metric parametrization

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

i.e. by letting $A(r) = 1/f(r)$.

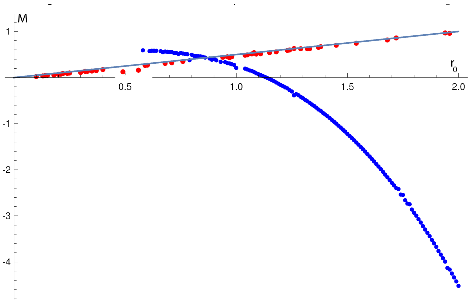
For $B(r)$ vanishing linearly in $r - r_0$ for some r_0, field-equation analysis shows that $f(r)$ must similarly be linearly vanishing at r_0, and accordingly one has a horizon. One can thus make near-horizon expansions (note $c = 1$ is arrangeable by $t \to c^{-1/2}t$)

$$\begin{aligned} B(r) &= c\left[(r-r_0) + h_2(r-r_0)^2 + h_3(r-r_0)^3 + \cdots\right] \\ f(r) &= f_1(r-r_0) + f_2(r-r_0)^2 + f_3(r-r_0)^3 + \cdots \end{aligned}$$

and the parameters h_i and f_i for $i \geq 2$ can then be solved-for in terms of r_0 and f_1. For the Schwarzschild solution, one has $f_1 = 1/r_0$, so it is convenient to parametrize the deviation from Schwarzschild using an NS parameter δ with

$$f_1 = \frac{1+\delta}{r_0}\,.$$

The task then becomes one of finding values of $\delta \neq 0$ for which the generic rising exponential behaviour as $r \to \infty$ is suppressed. What one finds is that there does indeed exist an asymptotically flat family of NS black holes which crosses the Schwarzschild family at a special horizon radius $r_0^{\rm Lich}$. For $\alpha = \frac{1}{2}, \gamma = 1$, one finds the following families of black holes:



Black-hole masses as a function of horizon radius r_0, with a crossing point at $r_0^{\rm Lich} \simeq$ 0.876. The red family denotes Schwarzschild black holes and the blue family denotes NS black holes.

The Lichnerowicz Operator

Now let us study in some more detail the point where the new black hole family crosses the classic Schwarzschild solution family. We can study solutions in the vicinity of the Schwarzschild family by looking at infinitesimal variations of the higher-derivative equations of motion around a Ricci-flat background. For the $\delta R_{\mu\nu}$ variation of the Ricci tensor away from a background with $R_{\mu\nu} = 0$, one obtains

$$\gamma(\delta R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\,\delta R) + 2(\beta - \tfrac{1}{3}\alpha)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\delta R$$
$$-2\alpha\Box\,(\delta R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\,\delta R) - 4\alpha\, R_{\mu\rho\nu\sigma}\,\delta R^{\rho\sigma} = 0\ .$$

Restricting attention to asymptotically flat solutions with horizons, however, we know from the trace no-hair theorem that $R = 0$ so $\delta R = 0$ and the $\delta R_{\mu\nu}$ equation simplifies, upon recalling that $m_2^2 = \frac{\gamma}{2\alpha}$, to

$$\left(\Delta_L + m_2^2\right) \delta R_{\mu\nu} = 0 \ ,$$

where the Lichnerowicz operator is given by

$$\Delta_L \, \delta R_{\mu\nu} \equiv -\Box \delta R_{\mu\nu} - 2 R_{\mu\rho\nu\sigma} \, \delta R^{\rho\sigma} \ .$$

Restricting attention to the $m_2^2 > 0$ nontachyonic case, one sees that black hole solutions deviating from Schwarzschild must have a $\lambda = -m_2^2$ negative Lichnerowicz eigenvalue for $\delta R_{\mu\nu}$.

The Gross-Perry-Yaffe eigenvalue

In a study of the thermodynamic instability of the Euclideanised Schwarzschild solution in Einstein theory, Gross, Perry and Yaffe Phys. Rev. D25 (1982), 330 found that there is just one normalisable negative-eigenvalue mode of the Lichnerowicz operator for deviations from the Schwarzschild solution. For a Schwarzschild solution of mass M, it is

$$\begin{aligned} \lambda &\simeq -0.192 M^{-2} \\ \textit{i.e.} \quad m_2 M &\simeq 0.438 \simeq \sqrt{.192} \end{aligned}$$

- Comparing with the numerical results for the new black hole solutions of the higher-derivative gravity theory, this corresponds nicely with the point where the NS black hole family crosses the Schwarzschild family.

Gregory-Laflamme Instability

The Gregory-Laflamme instability is an S-wave ($\ell = 0$) spherically symmetric instability, originally found in the context of 5D black strings.

In the higher-derivative theory, an analogous instability exists for low-mass Schwarzschild black holes, but it disappears for black hole masses $M \geq M_{\max}$ where

$$\frac{m_2 M_{\max}}{M_{\mathrm{Pl}}^2} = .438$$

This is precisely the crossing point between the family of NS black holes and the Schwarzschild family.

Note that the existence of this S-wave instability depends critically on the presence in the theory of the m_2 massive spin-two mode.

Thermodynamic Implications for Instability

The $D = 4$ Wald entropy formula

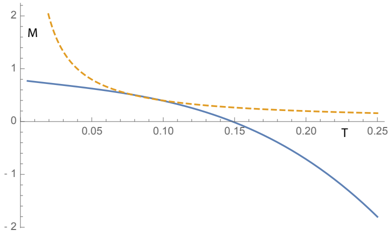
$$S = -\tfrac{1}{8} \int_{+} \sqrt{h} d\Sigma \, \epsilon^{ab} \epsilon^{cd} \frac{\partial L}{\partial R^{abcd}}$$

gives results that respect the first law of black-hole thermodynamics, $dM = TdS$.

For the NS black holes in $D = 4$, one obtains the following numerical relations between mass, temperature and entropy:

$$\begin{aligned} M_{\rm NS} &\approx 0.168 + 0.131\, S - 0.00749\, S^2 - 0.000139\, S^3 + \cdots \\ T_{\rm NS} &\approx 0.131 - 0.0151\, S - 0.000428\, S^2 + \cdots \end{aligned}$$

Recall that for Schwarzschild black holes, one has the classic mass-temperature relation $M_{\mathrm{Sch}} = \frac{1}{8\pi T}$. Eliminating the entropy for the NS black holes, one obtains the corresponding relations between black-hole mass and temperature for Schwarzschild and NS black holes:

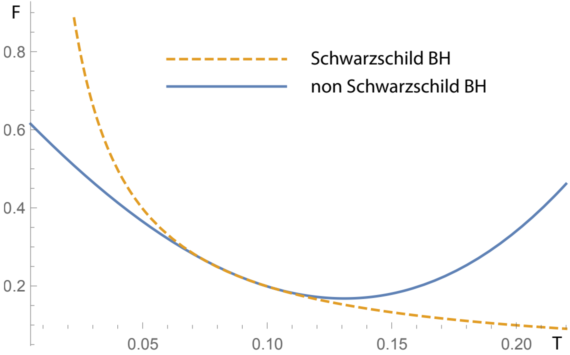


Mass versus temperature relations for Schwarzschild (dashed red) and non-Schwarzschild (solid blue) black holes.

Note that, when compared at the same mass M, the NS black holes are always colder than the Schwarzschild black holes (except at the Lichnerowicz point).

Free Energy

For the free energy $F = M - TS$, one has the following situation, implying a switchover in stability properties between the Schwarzschild and the NS solutions:



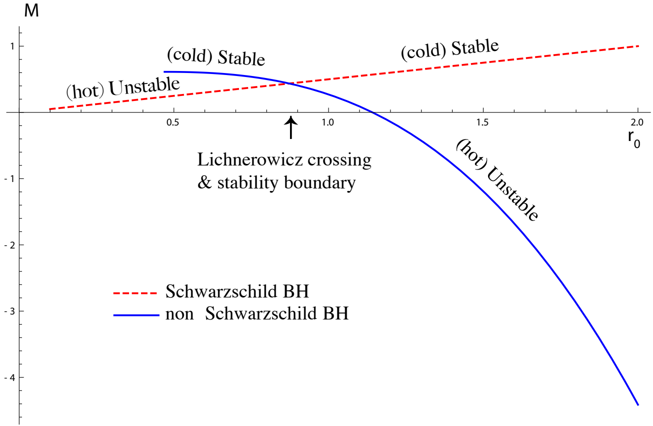
Free energy for Schwarzschild (dashed red) and non-Schwarzschild (blue) black holes. Lower free energy corresponds to greater stability.

Thermodynamic versus dynamical instabilities

Gubser and Mitra proposed a relationship between thermodynamic and dynamical instabilities: time-dependent dynamical instability cannot occur without a corresponding thermodynamic instability in the related finite-temperature Euclidean theory. JHEP 0108 (2001) 018

This has been proved in the context of axisymmetric black holes in Einstein theory by Hollands and Wald Commun.Math.Phys. 321 (2013) 629 .

Assuming the same relation holds between dynamical and thermodynamic instabilities in the higher-derivative gravity theory, and taking into account the known Gregory-Laflamme instability for Schwarzschild black holes below the Lichnerowicz crossing point, one obtains a clear suggestion for the respective domains of stability and instability of the Schwarzschild and the NS black holes.



Classical stability regimes. The dashed red line denotes Schwarzschild black holes and the solid blue line denotes non-Schwarzschild black holes.

Numerical study of non-Schwarzschild black holes

Owing to the highly nonlinear nature of the field equations, connecting the strong-field region near $r = 0$ to the weak-field $r \to \infty$ region of the NS black holes requires a careful numerical study which was made by Bonanno and Silveravalle.

Phys.Rev.D 99 (2019) 10, 101501; 1903.08759 [gr-qc]

Recall that for asymptotically flat solutions with a horizon, there is a trace no-hair theorem which makes it sufficient to study just the Einstein-Weyl theory $I = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$.

Assume the existence of a horizon at radius r_0 and asymptotic flatness, again writing the metric ansatz as $ds^2 = -B(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d^2\Omega$. The asymptotically flat linearised solution near the horizon has the form

$$\begin{aligned} B(r) &= h_1(r - r_0) + h_2(f_1, r_0)\,(r - r_0)^2 + \cdots (\text{convention: } h_1 = 1) \\ f(r) &= f_1\,(r - r_0) + f_2(f_1, r_0)\,(r - r_0)^2 + \cdots \end{aligned}$$

where $f_{i>1}$ and $h_{i>1}$ are determined by f_1 and r_0. The surface gravity is $\kappa = \frac{1}{2}\sqrt{f_1 h_1}$ and the temperature is $T = \frac{\kappa}{2\pi}$.

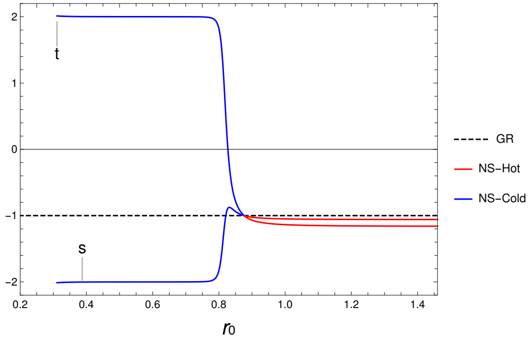
The form of an asymptotically flat linearised weak-field solution as $r \to \infty$ is characterised by two essential parameters, M and S^{2-}:

$$\begin{aligned} f(r) &= 1 - \frac{2M}{r} + S^{2-}\frac{e^{-m_2 r}}{r}(1 + m_2 r) \\ B(r) &= 1 - \frac{2M}{r} + 2S^{2-}\frac{e^{-m_2 r}}{r} \end{aligned}$$

Starting from a solution of this form at a radius $r \gg r_0$ and then numerically integrating inwards towards a fitting radius r_f, the task is then to use shooting methods to make at $r = r_f$ a match with a solution as expanded around a horizon at some radius r_0.

Having achieved such a match between the solution at infinity and at the horizon, the numerical integration can then be continued further inwards towards the origin at $r = 0$ for comparison with the solution families found by Frobenius analysis at the origin.

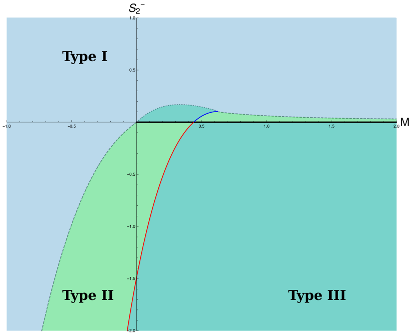
What is found is that there is a changeover in the near-origin behaviour, depending on the value of r_0. For definiteness, the analysis was performed with $m_2 = 1$. Black holes with $M > 0$ then only exist for $S^{2-} > -1.5$. NS black holes with $S^{2-} > 0$ are colder than Schwarzschild black holes with the same horizon radius, while NS black holes with $S^{2-} < 0$ are hotter.



Near-origin behaviour of NS black hole solutions. $t = r\partial_r \ln B(r)$, $s = r\partial_r \ln f(r)$

Recalling that $f(r) = \frac{1}{A(r)}$ (so $\tilde{s} = -s$), one thus finds a switchover in behaviour at around $r_0 \sim 0.86$ from NS-hot black holes with near-origin behaviour $(s, t) \sim (-1, -1)$ (as for Schwarzschild) to NS-cold $(-2, 2)$ solutions with a vanishing metric at the origin.

This study considered, by construction, solutions with horizons, so it was not going to find the $(-2, 2)$ horizonless solutions found by Bob Holdom. More general study of the $(M,\ S^{2-})$ phase space of solutions (Alun Perkins' PhD thesis; unpublished work of Bonanno and Silveravalle) shows that the horizon solutions are to be found on the boundaries between $(-2, 2)$ horizonless solutions and $(-1, -1)$ solutions or between $(-2, 2)$ horizonless solutions and wormholes.



NS Black hole phases in $R - C^2$ gravity (Courtesy Bonanno and Silveravalle)

- Type I: $(s, t) = (-1, -1)_0$ solutions singular at the origin
- Type II: $(-2, 2)_0$ solutions with vanishing metric at the origin
- Type III: $(1, 0)_{r_0}$ and other wormholes

Solutions with horizons live on the boundaries between the various Type I, II, III solutions. Schwarzschild solutions lie on the bold black line along the M axis.

Overvue

- The $I = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)$ theory has a richer static classical solution set than Einstein theory: in addition to the standard Einstein Ricci-flat static vacuum solutions, there are solutions without a horizon, wormhole solutions, and also a family of non-Schwarzschild black hole solutions.
- The Schwarzschild and non-Schwarzschild black-hole solution families cross at a mass $M_{\rm Lich}$ which is related to the Gross-Perry-Yaffe negative-eigenvalue mode λ of the Lichnerowicz operator by $\lambda = -m_2^2 \simeq -0.192 M_{\rm Lich}^{-2}$.
- The Schwarzschild solution family develops a Gregory-Laflamme S-wave instability for solutions with radii below a minimum radius $r_0^{\rm Lich} = 2M_{\rm Lich}$ while thermodynamic analysis implies that the non-Schwarzschild black holes are stable for solutions with radii below $r_0^{\rm Lich}$.