Cosmological Perturbations in Palatini Formalism

MASAHIDE YAMAGUCHI

(Tokyo Institute of Technology)

03/12/21@QUANTUM GRAVITY, HIGHER DERIVATIVES & NONLOCALITY

Mio Kubota, Kin-ya Oda, Keigo Shimada, MY:
arXiv 2010.07867 [hep-th], JCAP03(2021)006

$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

Contents

- **Introduction**

  What's Palatini formalism ? **c.f. Roberto Percacci's talk**

- **Palatini formalism and cosmological perturbations**

  Einstein gravity

  Non-minimal coupling case (L2, L4)

  Galileon (L3)

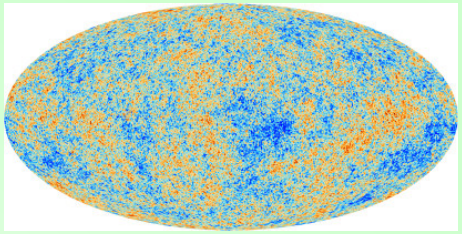
- **Discussion and conclusions**

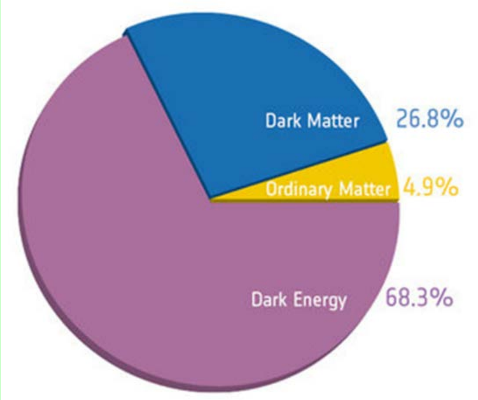
**(There are many many references related to Palatini formalim.
I omit most of them in this talk because of no enough space, sorry.
Please see the references in our paper and so on.)**

Introduction

Inflation & dark energy

Inflation & dark energy (cosmic acceleration in early and current Universe) are strongly supported by observations.





PLANCK

In order to identify the inflaton and the source of dark energy, it is quite useful to consider the most general models based on a scalar tensor theory.

Then, we can constrain models (or to single out the true model finally) from the observational results. (Bottom-up approach)

The following question arises:

What is the most general scalar-tensor theory?

(Bottom-up approach)

How widely can we extend scalar tensor theory ?

- **A kinetic term of an inflaton is not necessarily canonical.**

$$\mathcal{L} = X - V(\phi), \quad X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \quad \Longrightarrow \quad \mathcal{L} = K(\phi, X)$$

(k-inflation)

(Armendariz-Picon et.al. 1999)

- **An inflaton is not necessarily minimally coupled to gravity.**

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}M_G^2 R + \mathcal{L}_\phi\right) \quad \Longrightarrow \quad \Delta S = \int d^4x\sqrt{-g}f(\phi)R$$

(Brans-Dicke, Higgs inflation)

(Cervantes-Cota & Dehnen 1995, Bezrukov & M. Shaposhnikov 2008)

- **Action may include higher derivatives.** (Nicolis et.al. 2009)

$$\mathcal{L} = K(\phi, X) \quad \Longrightarrow \quad \Delta\mathcal{L} = G(\phi, X)\Box\phi$$

However, generally speaking, a higher derivative term is dangerous due to Ostrogradsky ghost.

Ostrogradski's theorem

(Ostrogradsky 1850)

Assume that $L = L(q, \dot{q}, \ddot{q})$ **and** $\frac{\partial L}{\partial \ddot{q}}$ **depends on** $\ddot{q}$ **:**

(Non-degeneracy)

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}^2 t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \quad \Longrightarrow \quad q^{(4)} = q^{(4)}\left(q^{(3)}, \ddot{q}, \dot{q}, q\right).$$

Canonical variables :

$$\begin{cases} q, & p := \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} \left(= \frac{\partial L_{\mathrm{eq}}}{\partial \dot{q}}\right), \\ Q := \dot{q}, & P := \frac{\partial L}{\partial \ddot{q}} \left(= \frac{\partial L_{\mathrm{eq}}}{\partial \dot{Q}}\right). \end{cases}$$

$$L_{\mathrm{eq}} = L(\dot{Q}, \dot{q}, q) + \lambda(Q - \dot{q})$$

Non-degeneracy $\Leftrightarrow \quad \ddot{q} = \ddot{q}\left(q, \dot{q}, \frac{\partial L}{\partial \ddot{q}}\right) \quad \Leftrightarrow \quad \dot{Q} = \ddot{q} = \ddot{q}(q, Q, P)$

Hamiltonian:

$$\begin{aligned} H(q, Q, p, P) &:= p\dot{q} + P\dot{Q} - L \\ &= pQ + P\,\ddot{q}(q, Q, P) - L(q, Q, \ddot{q}(q, Q, P)). \end{aligned}$$

p depends linearly on H so that no system of this form can be stable !!

N.B. $\frac{\partial L}{\partial \phi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \phi)}\right) + \partial_\mu \partial_\nu\left(\frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi)}\right) = 0.$ $\Rightarrow$ $\frac{i}{(p^2 + m_1^2)(p^2 + m_2^2)} = \frac{1}{m_2^2 - m_1^2}\left(\frac{i}{p^2 + m_1^2} - \frac{i}{p^2 + m_2^2}\right).$

(propagators)

Bottom up approach

- **Effective field theory approach :** **(Weinberg 2008, Cheung et al. 2008)**

  The low-energy effective theory (after integrating out heavy mode with its mass M).

  A ghost seems to appear around the cut-off scale M (>> E).

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\frac{1}{M^2}(\Box\phi)^2$$

$$\partial_\mu\phi\partial^\mu\phi \sim E^2\phi^2, \qquad \frac{1}{M^2}(\Box\phi)^2 \sim \frac{E^4}{M^2}\phi^2$$

(E: the energy scale we pay attention to)

$$M \gg E \quad \Longleftrightarrow \quad \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \gg \frac{1}{2}\frac{1}{M^2}(\Box\phi)^2$$

- **Most general theory without ghost**

  (if we are interested in the case in which higher derivative terms play an important role in the dynamics.)

In this talk, we take the latter approach

Integrating out a heavy field

σ : a heavy field with mass M, φ: a light field

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}M^2\sigma^2 - \partial_\mu\sigma\partial^\mu\phi$$

Lagrangian for σ

$$\wr$$

$$E^2\sigma^2 \ll M^2\sigma^2$$

$$\uparrow$$

energy scale we are interested in (E << M)

$$\sim -\frac{1}{2}M^2\sigma^2 + \sigma\Box\phi = -\frac{1}{2}M^2\left(\sigma - \frac{\Box\phi}{M^2}\right)^2 + \frac{1}{2}\frac{1}{M^2}(\Box\phi)^2$$

Integrating out σ

$$\sim \frac{1}{2}\frac{1}{M^2}(\Box\phi)^2$$

How to obtain a higher derivative theory without ghost ?

1. To abandon the non-degeneracy condition, which is assumed in the Ostrogradsky theorem.

2. To go into infinite derivative theories.

This talk

Galileon field (degenerate case)

Nicolis et al. 2009
Deffayet et al. 2009

The theory has Galilean shift symmetry in flat space :

$$\partial_\mu \phi \longrightarrow \partial_\mu \phi + b_\mu$$

$$\left\{ \begin{aligned} \mathcal{L}_1 &= \phi \\ \mathcal{L}_2 &= (\partial\phi)^2 \\ \mathcal{L}_3 &= (\partial\phi)^2 \,\Box\phi \\ \mathcal{L}_4 &= (\partial\phi)^2 \left[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2\right] \\ \mathcal{L}_5 &= (\partial\phi)^2 \left[(\Box\phi)^3 - 3\,(\Box\phi)\,(\partial_\mu\partial_\nu\phi)^2 + 2\,(\partial_\mu\partial_\nu\phi)^3\right] \end{aligned} \right.$$

$$(\partial_\mu\partial_\nu\phi)^2 = \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi,$$
$$(\partial_\mu\partial_\nu\phi)^3 = \partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi$$

Lagrangian has higher order derivatives, but EOM is second order.

→ **How to covariantize it ???**

Gravity

Two formalisms:
metric formalism & Palatini formalism

c.f. Roberto Percacci's talk

Metric formalism

Palatini formalism

Fundamental objects (dynamical variables)

Metric formalism

- **Riemann metric:** $g_{\mu\nu}$

Symmetric 2nd rank tensor determining the length

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- **Connection (parallel transport)**

Fixed a priori

→ **Levi-Civita connection :**

$$\left\{ {\lambda \atop \mu\nu} \right\}_g := \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

- **symmetric (** $\left\{ {\lambda \atop \mu\nu} \right\}_g = \left\{ {\lambda \atop \nu\mu} \right\}_g$ **)**
- **metric compatibility (** $\nabla_\lambda g_{\mu\nu} = 0$ **)**

←

- **Local Lorentz**
- **Invariance of an angle between parallel transported vectors.**

➔ **The variation of the action is taken only with respect to a metric in order to obtain the EOMs.**

Palatini formalism

- **Riemann metric:** $g_{\mu\nu}$

Symmetric 2nd rank tensor determining the length

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- **Connection:** $\Gamma^\lambda{}_{\nu\mu}$

(not confined to Levi-Civita one but arbitrary one)

- **Torsion :** $T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}$
- **Non-metricity :** $Q_\sigma{}^{\mu\nu} := \overset{\Gamma}{\nabla}_\sigma g^{\mu\nu}$

(In general, torsion does not vanish, but, for simplicity, we consider only a torsion-less case later.)

➔ **The variations of the action with respect to not only a metric but also a connection are taken in order to obtain the EOMs.**

Lesson:

What happens to the Einstein gravity in Palatini formalism ?

(Assume torsion-less)

Einstein gravity in Palatini formalism

(Einstein 1925)

$$S = S_{\rm EH} + S_{\rm matter} = \int d^4x \sqrt{-g}\,\frac{1}{2}\overset{\Gamma}{R} + \int d^4x \sqrt{-g}\,\mathcal{L}_{\rm m}(g_{\mu\nu}, \Psi).$$

(Assume no dependence on Γ)

$$\begin{cases} \overset{\Gamma}{R} := g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu}, \\ \overset{\Gamma}{R}_{\mu\nu} := \overset{\Gamma}{R}{}^{\lambda}{}_{\mu\lambda\nu}, \\ \overset{\Gamma}{R}{}^{\lambda}{}_{\sigma\mu\nu} := \partial_\mu \Gamma^{\lambda}{}_{\sigma\nu} - \partial_\nu \Gamma^{\lambda}{}_{\sigma\mu} + \Gamma^{\lambda}{}_{\rho\mu}\Gamma^{\rho}{}_{\sigma\nu} - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\rho}{}_{\sigma\mu}. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \overset{\Gamma}{R}_{(\mu\nu)} - \frac{1}{2}\overset{\Gamma}{R} g_{\mu\nu} - T_{\mu\nu} = 0, \\ \frac{\delta S}{\delta \Gamma^{\lambda}{}_{\nu\mu}} = \frac{1}{2}\overset{\Gamma}{\nabla}_\sigma \left[\sqrt{-g}\left(g^{\nu\sigma}\delta^{\mu}_{\lambda} - g^{\mu\nu}\delta^{\sigma}_{\lambda}\right)\right] = 0, \end{cases}$$

$$\left(T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}\right)$$

$$\Rightarrow \overset{\Gamma}{\nabla}_\lambda g_{\mu\nu} = 0 \Rightarrow \Gamma^{\lambda}{}_{\nu\mu} = \left\{{\lambda \atop \mu\nu}\right\}_g = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

Different from metric formalism, a connection is dynamically fixed to be the Levi-Civita connection as the result of the EOM.

Now, let's try to extend gravity to a scalar-tensor theory in Palatini formalism

But, before going to Palatini formalism, let's briefly remember a scalar-tensor theory in metric formalism.

Galileon field (degenerate case)

Nicolis et al. 2009
Deffayet et al. 2009

The theory has Galilean shift symmetry in flat space :

$$\partial_\mu \phi \longrightarrow \partial_\mu \phi + b_\mu$$

$$\begin{cases} \mathcal{L}_1 = \phi \\ \mathcal{L}_2 = (\partial\phi)^2 \\ \mathcal{L}_3 = (\partial\phi)^2 \Box\phi \\ \mathcal{L}_4 = (\partial\phi)^2 \left[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2\right] \\ \mathcal{L}_5 = (\partial\phi)^2 \left[(\Box\phi)^3 - 3(\Box\phi)(\partial_\mu\partial_\nu\phi)^2 + 2(\partial_\mu\partial_\nu\phi)^3\right] \end{cases}$$

$$(\partial_\mu\partial_\nu\phi)^2 = \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi,$$
$$(\partial_\mu\partial_\nu\phi)^3 = \partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi$$

Lagrangian has higher order derivatives, but EOM is second order.

➡ **How to covariantize this in metric formalims ?**

Generalized Galileon = Horndeski

Deffayet et al. 2009, 2011, Charmousis et al. 2012 Horndeski 1974

equivalence

Kobayashi, MY, Yokoyama 2011

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \qquad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \qquad G_{iX} \equiv \partial G_i/\partial X.$$

This is the most general scalar tensor theory whose Euler-Lagrange EOMs are up to second order though the action includes second derivatives.
Many of inflation and dark energy models can be understood in a unified manner.

NB :
- **$G_4 = M_G^2 / 2$ yields the Einstein-Hilbert action**
- **$G_4 = f(\varphi)$ yields a non-minimal coupling of the form $f(\varphi)R$**
- **The new Higgs inflation with $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ comes from $G_5 \propto \varphi$ after integration by parts.**

Horndeski theory

Horndeski 1974

In 1974, Horndeski presented the most general action (in four dimensions) constructed from the metric g, the scalar field φ, and their derivatives, $\partial g_{\mu\nu}, \partial^2 g_{\mu\nu}, \partial^3 g_{\mu\nu}, \cdots, \partial\phi, \partial^2\phi, \partial^3\phi, \cdots$ still having second-order equations.

$$\begin{aligned}\mathcal{L}_H \;=\; & \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left[\kappa_1 \nabla^\mu\nabla_\alpha\phi R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi + \kappa_3\nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\sigma} + 2\kappa_{3X}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\right] \\ & + \delta^{\alpha\beta}_{\mu\nu}\left[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2F_X\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi + 2\kappa_8\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\right] - 6\left(F_\phi + 2W_\phi - X\kappa_8\right)\Box\phi + \kappa_9.\end{aligned}$$

κ1, κ3, κ8, κ9, F : functions of φ & X with

W = W(φ)

$$\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\mu_1\mu_2\ldots\mu_n} = n!\,\delta^{[\alpha_1}_{\mu_1}\delta^{\alpha_2}_{\mu_2}\ldots\delta^{\alpha_n]}_{\mu_n}.$$

$$F_X = 2(\kappa_3 + 2X\kappa_{3X} - \kappa_{1\phi}).$$

What is the relation between Generalized Galileon and Horndeski's models ?

⇒ **Both models are completely equivalent :** **Kobayashi, MY, Yokoyama 2011**

$$\begin{cases} K = \kappa_9 + 4X\int^X dX'\left(\kappa_{8\phi} - 2\kappa_{3\phi\phi}\right), \\ G_3 = 6F_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2\int^X dX'(\kappa_8 - 2\kappa_{3\phi}), \\ G_4 = 2F - 4X\kappa_3, \\ G_5 = -4\kappa_1, \end{cases}$$

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

Cosmological perturbations of Horndeski theory in metric formalism

(Kobayashi, MY, Yokoyama 2011)

- **Tensor perturbations:**

$$S_T^{(2)} = \frac{1}{8}\int \mathrm{d}t\mathrm{d}^3x\, a^3\left[\mathcal{G}_T \dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\nabla h_{ij})^2\right].$$

$$\begin{cases} \mathcal{F}_T := 2\left[G_4 - X\left(\ddot{\phi}G_{5X} + G_{5\phi}\right)\right], \\ \mathcal{G}_T := 2\left[G_4 - 2XG_{4X} - X\left(H\dot{\phi}G_{5X} - G_{5\phi}\right)\right] \end{cases}$$

$$c_T^2 := \frac{\mathcal{F}_T}{\mathcal{G}_T}$$

If this Horndeski field is responsible for dark energy, the sound speed of tensor perturbations (GWs) must be very close to unity.

$$c_T^2 = c_{\mathrm{GW}}^2 \simeq 1.$$

(e.g. Creminelli & Vernizzi 2017)
(Kimura & Yamamoto 2012)

(GW170817 & GRB170817A)

(gravitational Cherenkov radiation)

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

→

$$G_{4X} \simeq 0, \quad G_5 \simeq 0$$

Let’s try to extend the flat Galileon action in Palatini formalism.

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \qquad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

$$\left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ G_{iX} \equiv \partial G_i/\partial X\right)$$

As a dark energy, only magenta boxes are allowed in metric formalism.

Horndeski correspondence in Palatini formalism

A non-minimal coupling of a scalar field to the Ricci scalar

In metric formalism,

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_4 = G_4(\phi, X) R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2\right]. \end{cases}$$

(Later, we will discuss L3)

$\left(c_T^2 \neq 1 \text{ for } G_{4X} \neq 0\right)$ $\left(X = -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ G_{iX} \equiv \partial G_i / \partial X\right)$

In Palatini formalism, $\mathcal{L}_4$ $\mathcal{L}_2$

$$S_4{}^{\text{Jordan}} := \int d^4x \sqrt{-g}\left[G_4(\phi, X) \overset{\Gamma}{R} + K(\phi, X)\right],$$

(The counter terms are unnecessary to keep the second order EOMs for the metric & φ.)

Analysis in three frames :

- **Jordan frame : Non-minimal coupling (Calculation is tedious but straightforward)**
- **Einstein frame : Minimal coupling, Einstein gravity (Calculation is well-known) (commonly used in the literatures, especially, in the context of Higgs inflation)**
- **Riemann frame : Geometry is Riemannian (Calculation is done in metric formalism)**

The central question : is c_T (GW speed) unity or not ?

Jordan frame

Connection in Jordan frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right], \qquad \left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\begin{cases} -2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = (K_X + G_{4X}\overset{\Gamma}{R})\partial_\mu\phi\partial_\nu\phi + (K + G_4\overset{\Gamma}{R})g_{\mu\nu} - 2G_4\overset{\Gamma}{R}_{(\mu\nu)} = 0. \\ \frac{\delta S}{\delta \Gamma^\lambda{}_{\nu\mu}} = \overset{\Gamma}{\nabla}_\sigma\left[\sqrt{-g}\,G_4\,(g^{\nu\sigma}\delta^\mu{}_\lambda - g^{\mu\nu}\delta^\sigma_\lambda)\right] = 0. \end{cases}$$

$$\overset{\Gamma}{\nabla}_\lambda g_{\mu\nu} = g^{\mu\nu}\partial_\lambda(\ln G_4) \neq 0.$$

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}_g + \frac{1}{2}g^{\lambda\sigma}\left(2g_{\sigma(\mu}\partial_{\nu)}\ln G_4 - g_{\mu\nu}\partial_\sigma \ln G_4\right).$$

The connection does not coincide with the Levi-Civita one in general.

Cosmological perturbations in Jordan frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

Metric perturbations : $ds^2 = -N^2dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$

$$\begin{cases} N &= 1+\alpha, \\ N_i &= \partial_i\beta, \\ \gamma_{ij} &= a(t)^2 e^{-2\zeta}\left(e^h\right)_{ij}. \end{cases}$$

(unitary gauge ←→ δφ = 0)

3 scalar perturbations : α, β, ζ
1(x2) tensor perturbations : hij

Connection perturbations :

$$\begin{cases} \delta\Gamma^0{}_{00} &= c_1, \\ \delta\Gamma^0{}_{i0} &= \partial_i c_2, \\ \delta\Gamma^0{}_{ij} &= D_{1,ij} + \delta_{ij}c_3 + \partial_i\partial_j c_4, \\ \delta\Gamma^i{}_{00} &= \partial^i c_5 \\ \delta\Gamma^i{}_{j0} &= D^i_{2,j} + \delta^i_j c_6 + \partial^i\partial_j c_7, \\ \delta\Gamma^i{}_{jk} &= \partial^i D_{3,jk} + \partial_{(j}D^i_{4,k)} + \delta_{jk}\partial^i c_8 + \delta^i_{(j}\partial_{k)}c_9 + \partial^i\partial_j\partial_k c_{10}. \end{cases}$$

10 scalar perturbations : cn
4(x2) tensor perturbations : Dm,ij

Cosmological perturbations in Jordan frame II

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

➡ **Expand the action up to quadratic order of perturbations**

➡ **Solve the constraints for lapse α, shift β, and connections**

➡

$$\delta^{(2)}S_4{}^{\text{Jordan,tensor}} = \frac{1}{4}\int dt d^3x G_4 a^3\left[\dot{h}_{ij}^2 - \frac{1}{a^2}(\partial_k h_{ij})^2\right]$$

➡ **$c_T = 1$ (GW speed = light speed)**

$$\delta^{(2)}S_4{}^{\text{Jordan,scalar}} = \int dt\, d^3x\, a^3\left[\mathcal{G}_S\dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\partial_i\zeta)^2\right]$$

(background quantities)

$$\mathcal{F}_S = \frac{6XG_4(2KG_{4X} - K_XG_4)}{(K - 2XK_X)G_4 + 3KG_{4X}X},$$

$$\mathcal{G}_S = \frac{6XG_4}{(G_4 - G_{4X}X)^2\{-K(G_4 + 3G_{4X}) + 2XK_XG_4\}} \times\Big[-6X^2KG_{4X}^3 + X(8K + 5K_XX)G_4G_{4X}^2 + (K_X + 2K_{XX}X)G_4^3 - 2\{K(G_{4X} + 2G_{4XX}X) + XK_X(3G_{4X} - XG_{4XX}) + X^2K_{XX}G_{4X}\}G_4^2\Big],$$

Einstein frame

Analysis in Einstein frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi,X)\overset{\Gamma}{R} + K(\phi,X)\right], \quad \left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

Conformal transformation : $\tilde{g}_{\mu\nu} = G_4(\phi,X)g_{\mu\nu}$

$$\left(\sqrt{-\tilde{g}} = G_4^2\sqrt{-g}, \quad \overset{\Gamma}{R} = g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} = G_4\tilde{g}^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} = G_4\overset{\Gamma}{\tilde{R}}, \quad X = G_4\tilde{X}, \quad \tilde{X} = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\Rightarrow \quad S_4{}^{\text{Jordan}} = \int d^4x\sqrt{-\tilde{g}}\left[\tilde{g}^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + \frac{K(\phi,G_4\tilde{X})}{G_4^2(\phi,G_4\tilde{X})}\right] := S_4{}^{\text{Einstein}}$$

This action is nothing but the k-essence action
and the Einstein-Hilbert action with respect to $\tilde{g}_{\mu\nu} = G_4(\phi,X)g_{\mu\nu}$ **.**

$$\Rightarrow \quad \Gamma^\lambda{}_{\nu\mu} = \left\{ {\lambda \atop \mu\nu} \right\}_{\tilde{g}} = \frac{1}{2}\tilde{g}^{\lambda\sigma}\left(\partial_\mu\tilde{g}_{\nu\sigma} + \partial_\nu\tilde{g}_{\mu\sigma} - \partial_\sigma\tilde{g}_{\mu\nu}\right)$$ **: Levi-Civita one with respect to** $\tilde{g}_{\mu\nu}$**.**

$$\Rightarrow \begin{cases} \delta^{(2)}S_4{}^{\text{Einstein,tensor}} = \frac{1}{4}\int d\tilde{t}\, d^3\tilde{x}\,\tilde{a}^3\left[\tilde{h}'^2_{ij} - \frac{1}{\tilde{a}^2}(\tilde{\partial}_k\tilde{h}_{ij})^2\right] \quad \left(= \delta^{(2)}S_4{}^{\text{Jordan,tensor}}\right) \\ \delta^{(2)}S_4{}^{\text{Einstein,scalar}} = \int d\tilde{t}\, d^3\tilde{x}\,\tilde{a}^3\left[\tilde{\mathcal{G}}_S\zeta'^2 - \frac{\tilde{\mathcal{F}}_S}{\tilde{a}^2}(\tilde{\partial}_i\tilde{\zeta})^2\right] \quad \left(= \delta^{(2)}S_4{}^{\text{Jordan,scalar}}\right) \end{cases}$$

$c_T = 1$!!

$$\left(\mathcal{G}_S = G_4(t)\tilde{\mathcal{G}}_S, \quad \mathcal{F}_S = G_4(t)\tilde{\mathcal{F}}_S\right) \qquad \left(h_{ij} = \tilde{h}_{ij}, \quad \zeta = \tilde{\zeta}\right)$$

Riemann frame

Analysis in Riemann frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right], \qquad \left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\Rightarrow \quad \Gamma^\lambda{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\}_g + \frac{1}{2}g^{\lambda\sigma}\left(2g_{\sigma(\mu}\partial_{\nu)}\ln G_4 - g_{\mu\nu}\partial_\sigma \ln G_4\right).$$

$$\left(\Rightarrow \quad \overset{\Gamma}{R} = \overset{g}{R} - \overset{g}{\nabla}_\sigma\left(\partial_\sigma \log G_4\right) - \frac{3}{2}g^{\mu\nu}\left(\partial_\mu \log G_4\right)\left(\partial_\nu \log G_4\right), \right)$$

$$\Rightarrow \quad S_4{}^{\text{Jordan}} = \sqrt{-g}\left[G_4\overset{g}{R} + \frac{3}{2}\frac{(\overset{g}{\nabla}G_4)^2}{G_4} + K\right]$$

$$= \sqrt{-g}\left[G_4\overset{g}{R} - \frac{3}{2G_4}\left(2G_{4\phi}^2X + 2G_{4\phi}G_{4X}\phi^\alpha\phi_{\alpha\beta}\phi^\beta - G_{4X}^2\phi^\alpha\phi_{\alpha\beta}\phi^{\beta\gamma}\phi_\gamma\right) + K\right] := S_4{}^{\text{Riemann}}$$

In this frame, the connection is a priori fixed to the Levi-Civita one.

$\Rightarrow$ **But, this is nothing but simple rewriting and hence both g and φ obey the same EOMs as those in Jordan frame.**

(Langlois & Noui 2016, Crisostomi et al. 2016, Ben Achour et al. 2016 …)

In fact, this action reduces to the so-called DHOST action and the quadratic actions for perturbations are shown to coincide with those in Jordan frame.

Cosmological perturbations in three frames

- **The quadratic actions for tensor and scalar perturbations in three different frames (Einstein, Jordan, Riemann) are obtained and also shown to be the same.**

- **Even if G4 has X-dependence, the speed of GWs is unity, in sharp contrast with the case of metric formalism.**

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

Let's finally discuss L3 (Galileon) action in Palatini formalism.

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

$$\left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ G_{iX} \equiv \partial G_i/\partial X\right)$$

As a dark energy, red crossed parts are prohibited in metric formalism.

L3 term (KGB or G-inflation) in metric formalism

(Kobayashi, MY, Yokoyama 2010, 2011
Cedric, Pujolas, Sawicki, Vikman 2010)

- The L3 term is **uniquely** determined in **metric** formalism

$$\overset{g}{\Box}\phi = g^{\mu\nu}\overset{g}{\nabla}_\mu\overset{g}{\nabla}_\nu\phi$$
$$= \overset{g}{\nabla}_\mu(g^{\mu\nu}\overset{g}{\nabla}_\nu\phi)$$
$$= \overset{g}{\nabla}_\mu\left(\overset{g}{\nabla}_\nu(g^{\mu\nu}\phi)\right)$$
$$= \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\overset{g}{\nabla}_\mu\left(g_{\alpha\beta}\overset{g}{\nabla}_\nu\phi\right)$$
$$\ldots$$

$$\begin{cases}\mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \qquad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].\end{cases}$$

All of these expressions are the same thanks to the metricity.

$$\left(\overset{g}{\nabla}_\lambda g_{\mu\nu} = 0\right)$$

- The L3 term does **not** affect the speed of GWs at all in **metric formalism**.

Tensor perturbations: $S_T^{(2)} = \frac{1}{8}\int dt d^3x\, a^3\left[\mathcal{G}_T\dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\nabla h_{ij})^2\right].$

$$\begin{cases}\mathcal{F}_T := 2\left[G_4 - X\left(\ddot{\phi}G_{5X} + G_{5\phi}\right)\right], \\ \mathcal{G}_T := 2\left[G_4 - 2XG_{4X} - X\left(H\dot{\phi}G_{5X} - G_{5\phi}\right)\right]\end{cases}$$

$$c_T^2 := \frac{\mathcal{F}_T}{\mathcal{G}_T}$$

L3 term (KGB or G-inflation) in Palatini formalism

- **The L3 term is not uniquely determined in Palatini formalism**

$$\overset{\Gamma}{\Box}\phi \overset{?}{=} g^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi \overset{?}{=} \overset{\Gamma}{\nabla}_\mu(g^{\mu\nu}\overset{\Gamma}{\nabla}_\nu\phi) \overset{?}{=} \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu(g^{\mu\nu}\phi) \cdots$$

g

$\nabla \quad \nabla \quad \phi$

Non-metricity : $Q_\lambda{}^{\mu\nu} := \overset{\Gamma}{\nabla}_\lambda g^{\mu\nu} \neq 0$

Fortunately, there are only finite (10) number of types.

$$\overset{\Gamma}{\Box}\phi = \begin{cases} g \times \nabla\nabla\phi & \text{1 term} \\ \nabla g \times \nabla\phi & \text{2 terms} \\ \nabla g \times \nabla g \times \phi & \text{5 terms} \\ \nabla\nabla g \times \phi & \text{2 terms} \end{cases}$$

(Aoki & Shimada 2018, 2019 Helpin & Volkov 2019, 2020)

$$\begin{aligned} \mathcal{L}_3{}^{\text{Palatini}} := \; & G_{3,0}\overset{g}{\Box}\phi + G_{3,1}Q^\mu\partial_\mu\phi + G_{3,2}\bar{Q}^\mu\partial_\mu\phi \\ & + G_{3,3}\phi Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + G_{3,4}\phi Q_{\alpha\beta\gamma}Q^{\beta\gamma\alpha} + G_{3,5}\phi Q^\mu Q_\mu + G_{3,6}\phi Q_\mu\bar{Q}^\mu \\ & + G_{3,7}\phi\bar{Q}_\mu\bar{Q}^\mu + G_{3,8}\phi\overset{g}{\nabla}_\mu Q^\mu + G_{3,9}\phi\overset{g}{\nabla}_\mu\bar{Q}^\mu \\ =: \; & \sum_{i=0}^{9} G_{3,i}\,\overset{\Gamma}{\Box}_{(i)}\phi \end{aligned}$$

$$\left(Q_\mu := \frac{1}{4}Q_{\mu\nu}{}^\nu,\ \bar{Q}^\mu := Q_\nu{}^{\nu\mu}\right)$$

$$(\ G_{3,i} = G_{3,i}(\phi, X)\ )$$

L3 term (KGB or G-inflation) in Palatini formalism II

$$\mathcal{L}_{2+3+4}^{\text{Palatini}} := K + \sum_{i=0}^{9} G_{3,i} \overset{\Gamma}{\Box}_{(i)}\phi + G_4 \overset{\Gamma}{R}$$

This action has, in general,

- **no Einstein frame**
- **an (Ostrogradsky) ghost mode**
- **non-unity sound speed of GWs.**

If we remove the ghost by suitable choice of G3,i ,
this model reduces to the DHOST model in metric formalism
with the sound speed of GWs being unity.
(This sound speed is the same with the correspondence
in metric formalism).

Summary

- We considered Palatini formalism, where the variation of an action is taken with respect to not only metric but also connection.
- We considered the case of a non-minimal coupling of a scalar field to the Ricci scalar (L4) plus k-essence action (L2) and discussed cosmological perturbations, yielding their quadratic actions in three different frames.
- The sound speed of GWs is always unity in Palatini formalism even if G4 includes X-dependence, in sharp contrast with that in metric formalism.
- We classified the Galileon action (L3) in Palatini formalism and found that there are essentially 10 different terms.
- An action consisting of these terms as well as L2+L4 generally leads to a ghost d.o.f. and the deviation from unity of the sound speed of GWs. However, once we eliminate such a ghost, the sound speed of GWs becomes unity, which coincides with that in metric formalism.